Maths for Computing
Tutorial 8

1. Give a combinatorial proof of

$$n! = \binom{n}{0}D(n) + \binom{n}{1}D(n-1) + \binom{n}{2}D(n-2) + \ldots + \binom{n}{n}D(0)$$

2. How many permutations of $\{1,2,\ldots,n\}$ exist such that none of them contains $(i, i+1)$ for $i \in \{1,2,\ldots,n-1\}$?

3. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

4. The number of all set partitions of $[n]$ into nonempty parts is denoted by $B(n)$, and is called the n th Bell number. We set $B(0) = 1$. Clearly, $B(n) = \sum_{i=0}^{n} S(n, i)$. Prove combinatorially that

$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$.

5. Prove that if $n \geq 2$, then $n! < S(2n, n) < (2n)!$.

6. Prove that the number of partitions of n into at most k parts is equal to the number of partitions of $n + k$ into exactly k parts. (Hint: Use Ferrer diagrams.)

7. Prove that for all integers $n > 2$, the number $p(n) - p(n-1)$ is equal to the number of partitions of n in which the two largest parts are equal.

8. Prove that $p_3(n) = |X|$, where X is the set of 3 length partitions of $2n$, where every element of the partition is at most $n - 1$.

9. Prove that $C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}$, where C_n is the nth Catalan number for $n \geq 1$. (Hint: Use North-East lattice paths.)

10. Prove that the number of ways in which an n-sided polygon can be triangulated is equal to $(n-2)$th Catalan number. (Hint: Use recurrence relation in the previous problem)